Hindawi Publishing Corporation
Boundary Value Problems
Volume 2010, Article ID 147301, 17 pages
doi:10.1155/2010/147301

Research Article

On the Time Periodic Free Boundary Associated to Some Nonlinear Parabolic Equations

M. Badii[1] and J. I. Díaz[2]

[1] *Dipartimento di Matematica G. Castelnuovo, Università degli Studi di Roma "La Sapienza", P.le A. Moro 2, 00185 Roma, Italy*
[2] *Departamento de Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de las Ciencias, 3, 28040 Madrid, Spain*

Correspondence should be addressed to J. I. Díaz, ildefonso.diaz@mat.ucm.es

Received 30 July 2010; Accepted 1 November 2010

Academic Editor: Vicentiu Radulescu

We give sufficient conditions, being also necessary in many cases, for the existence of a periodic free boundary generated as the boundary of the support of the periodic solution of a general class of nonlinear parabolic equations. We show some qualitative properties of this free boundary. In some cases it may have some vertical shape (linking the free boundaries of two stationary solutions), and, under the assumption of a strong absorption, it may have several periodic connected components.

1. Introduction

This paper deals with several qualitative properties of the time periodic free boundary generated by the solution of a general class of second-order quasilinear equations. To simplify the exposition we will fix our attention in the problem formulated on the following terms:

$$\begin{aligned} &u_t - \Delta_p u + \lambda f(u) = g \quad \text{in } Q := \Omega \times \mathbb{R}, \\ &u(x,t) = h(x,t) \quad \text{on } \Sigma := \partial\Omega \times \mathbb{R}, \\ &u(x,t+T) = u(x,t) \quad \text{in } Q. \end{aligned} \tag{P}$$

Here $T > 0$, $\Omega \subset \mathbb{R}^N (N \geqslant 1)$ denotes an open bounded and regular set, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$ is the so-called p-Laplacian operator, λ is a positive parameter, and the data f,g, and h are assumed to satisfy the following structural assumptions:

(H_f): $f \in C(\mathbb{R})$ is a nondecreasing function, $f(0) = 0$ and there exist two nondecreasing continuous functions f_1, f_2 such that $f_2(0) = f_1(0) = 0$ and

$$f_2(s) \leqslant f(s) \leqslant f_1(s), \quad \forall s \in \mathbb{R}, \tag{1.1}$$

(H_g): $g \in C(\mathbb{R}; L^\infty(\Omega))$ and g is T-periodic,

(H_h): $h \in C(\Sigma)$ and h is T-periodic.

We point out that our results remain true under a great generality (e.g., f can be replaced by a maximal monotone graph of $\mathbb{R}^2$, function g can be assumed merely in $C(\mathbb{R}; L^1(\Omega) + W^{-1,p'}(\Omega))$, and h can be assumed in a suitable trace space); nevertheless we prefer this simple setting to avoid technical aspects. In fact, most of the qualitative results of this paper remain valid for the more general formulation

$$\begin{gathered} b(u)_t - \Delta_p u + \lambda f(u) = g \quad \text{in } Q := \Omega \times \mathbb{R}, \\ u(x,t) = h(x,t) \quad \text{on } \Sigma := \partial\Omega \times \mathbb{R}, \\ u(x,t+T) = u(x,t) \quad \text{in } Q, \end{gathered} \tag{P_b}$$

when $b \in C(\mathbb{R})$ is a nondecreasing function such that $b(0) = 0$ but again we prefer to restrict ourselves to the special case of problem (P) (i.e., problem (P_b) with $b(u) = u$) to simplify the exposition. Notice, in particular, that the associated stationary equations have a common formulation (use $b(u)$ as new unknown in the case of problem (P_b)). We also recall that for $p = 2$ the diffusion operator becomes the usual Laplacian operator. Problems of this type arise in many different applications (see, e.g., [1, 2] and their references).

Many results on the existence and uniqueness of (weak) periodic solutions are already available in the literature (see the biographical comments collected in Section 1). Nevertheless those interesting questions are not our main aim here but only the study of the free boundary generated by the solution under suitable additional conditions on the data.

As in [1], given a function $\varphi : Q \to \mathbb{R}$, $\varphi \in C([0,T] : L^1_{\text{loc}}(\Omega))$, we will denote by $S(\varphi(\cdot,t))$ the subset of $\overline{\Omega}$ given by the support of the function $\varphi(\cdot,t)$, for any fixed $t \in \mathbb{R}$, and by $N(\varphi(\cdot,t))$ to the null set of $\varphi(\cdot,t)$ defined through $\overline{\Omega} - S(\varphi(\cdot,t))$. Sometimes this set is called as the *dead core* of φ in the framework of chemical reactions [1]. The boundary of the set

$$\cup_{t\in\mathbb{R}} N(\varphi(\cdot,t)) \tag{1.2}$$

is a *free boundary* in the case in which φ is the actual solution of problem (P) (or (P_b)): its existence and location are not a part of the *a priori* given formulation of the problem. For instance, in the context of chemical reactions, the formation of a dead core arises when the diffusion process is not strongly fast enough or equivalently the reaction term is very strong as to draw the concentration of reactant from the boundary into the central part of Ω (see, e.g., [1, 3, 4], among many other possible references). In the context of filtration in porous media (case of problem (P_b)) the formation of the free boundary is associated to the slow diffusion obtained through the Darcy law (see, e.g., [2] and its many references).

We point out that some important differences appear between the case of time periodic auxiliary conditions and the case of the usual initial boundary value problem when studying the formation of the free boundary. For instance, if we assume that there is no absorption term ($f(s) \equiv 0$), it is well known [2] that for the initial boundary value problem the formation of the free boundary is assured if $p > 2$ (or when $m(p-1) > 1$, for the case of problem (P_b) with $b(u) = |u|^{1/m-1}u$). But this cannot be true for the case of periodic conditions since it is well known that, for the case of nonnegative solutions, if $u(x_0, t_0) > 0$ then $u(x_0, t) > 0$ for any $t \geqslant t_0$ (see, e.g., [5] for the case of problem (P_b)). This property holds also in the presence of some additional transport terms (typical of filtration in porous media models), and so the time periodic solution does not generate any free boundary (as it is the case of the formulation considered in [6]).

In Section 2 we will obtain some sufficient conditions for the formation of a time periodic free boundary (which are also necessary in some sense) according the nature of the auxiliary functions $f_i(s)$, $g_i(x)$ and $h_i(x)$, $i = 1, 2$, involved in the structural assumptions (H_f), (H_g) and (H_h).

In Section 3 we will prove that if the data $g(x,t)$ and $h(x,t)$ become time independent during some subintervals (let us say on an interval $[t_1, t_2] \subset [0, T]$), then it is possible to construct some periodic solutions which become time independent (and so its associated free boundary) on some nonvoid subinterval of $[t_1, t_2]$. This qualitative property, which, at the best of our knowledge, is proved here for first time in the literature, implies that the free boundary may have vertical tracts linking the free boundaries of two stationary solutions. Finally, under the additional assumption of a strong absorption, we show that this free boundary may have several periodic connected components.

2. Sufficient Conditions for the Existence of the Periodic Free Boundary

Together with problem (P) we consider the following stationary problems:

$$\begin{gathered} -\Delta_p v + \lambda f_1(v) = g_1 \quad \text{in } \Omega, \\ v = h_1 \quad \text{on } \partial\Omega, \end{gathered} \tag{$\underline{SP}$}$$

$$\begin{gathered} -\Delta_p w + \lambda f_2(w) = g_2 \quad \text{in } \Omega, \\ w = h_2 \quad \text{on } \partial\Omega, \end{gathered} \tag{$\overline{SP}$}$$

where the data are now the auxiliary functions $f_i(s)$, $g_i(x)$, and $h_i(x)$, $i = 1, 2$, involved in the structural assumptions (H_f), (H_g), and (H_h). More precisely, assumptions (H_g) and (H_h) imply the existence of two bounded functions g_1, g_2 and two continuous functions h_1, h_2 such that

$$\begin{gathered} g_1(x) \leqslant g(x,t) \leqslant g_2(x), \quad \forall t \in \mathbb{R}, \quad \text{a.e. } x \in \Omega, \\ h_1(x) \leqslant h(x,t) \leqslant h_2(x), \quad \forall (x,t) \in \Sigma. \end{gathered} \tag{2.1}$$

We recall that by well-known results, problems ($\underline{SP}$) and ($\overline{SP}$) have a unique solution u_1, $u_2 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ (see, e.g., [1]). Concerning the existence, uniqueness and comparison principle of periodic solutions of problems (P) and (P_b), and other related problems, we restrict ourselves to present here some bibliographic remarks. As indicated before, those questions are not the main aim of this paper but the study of the free boundary generated by the solution under suitable additional conditions on the data.

There are many papers in the literature concerning the existence and uniqueness of a periodic solution of problems (P) (resp. (P_b)) under different assumptions on the data f, g, and h (resp. b). Perhaps one of the more natural arguments to get the existence of time periodic solutions of problems of this type is to show the existence of a fixed point for the Poincaré map. This was made already in [7] and by many other authors for the case of semilinear parabolic problems. One of the most delicate points in this method, especially when the parabolic problem becomes degenerate or singular, is to show the compactness of the Poincaré map. Sometimes this compactness argument comes from nontrivial regularity results of some auxiliary problems (see, e.g., [6, 8]). In some other cases it is used the compactness of the Green type operator associated to the semigroup generated by the diffusion operator [9, 10]. This can be proved also for doubly nonlinear diffusion operators like in problem (P_b) in the framework of variational periodical solutions $W_{T\text{-per}} := \{u - h \in L^p(0,T;W_0^{1,p}(\Omega)), u_t - h_t \in L^q(0,T;W^{-1,p'}(\Omega))$, and $u(\cdot, t+T) = u(\cdot, t)\ \forall t \in \mathbb{R}\}$ (observe that $W_{T\text{-per}} \subset C([0,T] : L^p(\Omega))$. Among the many references in the literature we can mention, for instance, [11–15] and references therein. For periodic solutions in the framework of *Alt-Luckaus type weak solutions* see, for instance, [16, 17]. The presence of some nonlinear transport terms require sometimes an special attention ([6, 18] and references therein).

The monotone and accretive operators theory leads to very general existence and uniqueness results on time periodic solutions of dissipative type problems. See, for instance, [19–27], and their many references. The abstract results lead to some perturbation results which apply to some semilinear problems [28, 29]. The case of superlinear semilinear equations was considered by several authors in [30] and references therein.

The existence of periodic solutions can be obtained also outside of a variational framework, for instance, when the data are merely in $L^1(\Omega)$ or even Radon measures. An abstract result in general Banach spaces (with important applications to the case of $L^1(\Omega)$ was given in [23]. For the case of Radon measures, see [31]. The case of variational inequalities and multivalued representations of the term $f(u)$ was considered in [32]. Different boundary conditions were considered in [33–35] and references therein. The case of a dynamic boundary condition was considered in [36]. For a problem which is not in divergence form, see [37].

The monotonicity assumptions imply the comparison principle and then the uniqueness of periodic solution ([6] and references therein) and the continuous dependence with respect to the data ([12] and references therein). Nonmonotone assumptions, especially on the zero-order term $f(u)$, originate multiplicity of solutions ([25, 38, 39] and references therein). Sometimes the method of super and subsolution can be applied by passing through an auxiliary monotone framework and applying some iterating arguments ([34, 40, 41], and references therein). This applies also to the case in which $f(u)$ can be singular [42].

We end this list of biographical comments by pointing out that the literature on the existence of periodic solutions for coupled systems of equations is also very large since many points of view have been developed according the peculiarities of the involved systems. A deep result on reaction diffusion systems can be found in [43]. For instance, the case of the *thermistor system* was the main goal of a series of papers by Badii [44–47].

Now we return to the study of the formation of a periodic free boundary. As mentioned before, under the monotonicity assumptions (H_f), it is easy to prove the existence and uniqueness of a (weak) solution of problem (P) as well as the following comparison result.

Lemma 2.1. *Assume (H_f), (H_g), and (H_h). Let $u(x,t)$ be the unique periodic solution of problem (P). Then*

$$u_1(x) \leqslant u(x,t) \leqslant u_2(x), \quad \forall t \in \mathbb{R}, \quad a.e.\ x \in \Omega. \tag{2.2}$$

As a consequence of Lemma 2.1 we have the following.

Corollary 2.2. *Assume (H_f), (H_g), and (H_h). Then one has the following.*

(i) *If $g_1, h_1 \geqslant 0$, then $N(u_1) \supset N(u(\cdot,t)) \supset N(u_2)$ $\forall t \in \mathbb{R}$. Analogously, if $g_2, h_2 \leqslant 0$ then $N(u_1) \subset N(u(\cdot,t)) \subset N(u_2)$for all $t \in \mathbb{R}$.*

(ii) *If $g_1, h_1 \geqslant 0$ and $u_1(x) > 0$ in Ω, then $u(x,t) > 0$ for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$. Analogously, if $g_2, h_2 \leqslant 0$ and $u_2(x) < 0$ in Ω, then $u(x,t) < 0$ for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$.*

In consequence, the existence of a periodic free boundary for problem (P) is implied by the existence of a free boundary for the auxiliary stationary problems. As indicated in [1], the existence of a free boundary for the stationary problems $(\underline{SP})$ and $(\overline{SP})$ (the free boundary is given as the boundary of the support of the solution) requires two types of conditions: (a) *a suitable balance between the diffusion and the absorption terms* and (b) *a suitable balance between "the size" of the null set of the data $N(h_i) \cup N(g_i)$ and "the size" of the solution* (e.g., its L^∞ -norm when it is bounded). A particular statement on the existence (and nonexistence) of a periodic free boundary is the following.

Theorem 2.3. *Assume (H_f), (H_g), (H_h), and let $g_1, h_1 \geqslant 0$. Let $F_i(s) = \int_0^s f_i(s)ds$, and assume that*

$$\int_{0^+} \frac{ds}{F_i(s)^{1/p}} < +\infty, \quad i = 1, 2. \tag{2.3}$$

Then, if $u(x,t)$ denotes the unique periodic solution of problem (P), one has that $N(u_1) \supset N(u(\cdot,t)) \supset N(u_2)$ for all $t \in \mathbb{R}$. In particular, $N(u(\cdot,t))$ contains, at least, the set of $x \in N(h_2) \cup N(g_2)$ such that

$$d\big(x, \partial\big(N(h_2) \cup N(g_2)\big)\big) > \Psi_{2,N}\Big(\|u_2\|_{L^\infty(\Omega)}\Big), \tag{2.4}$$

where

$$\Psi_{2,N}(\tau) = \left(\frac{N(p-1)}{p}\right)^{1/p} \int_0^\tau \frac{ds}{F_2(s)^{1/p}}. \tag{2.5}$$

Nevertheless, if $\min_{\partial\Omega} h_1 \geqslant k > 0$ *and if*

$$R < \Psi_{1,1}(k), \tag{2.6}$$

then $N(u(\cdot,t))$ *is empty since one has* $0 < u_1(x) \leqslant u(x,t)$ *for all* $t \in \mathbb{R}$ *and a.e.* $x \in \Omega$*. Here* R *is the radius of the smaller ball containing* Ω *and*

$$\Psi_{1,1}(\tau) = \left(\frac{(p-1)}{p}\right)^{1/p} \int_0^\tau \frac{ds}{F_1(s)^{1/p}}. \tag{2.7}$$

The proof is a direct consequence of [1, Corollary 1 and Theorem 1.9 and Proposition 1.22]. Many other results available for the auxiliary stationary problems lead to similar answers for the periodic problem (P). For instance we have the following.

Theorem 2.4. *Under assumptions* (H_f), (H_g), *and* (H_h), *if* $g_1, h_1 \geqslant 0$ *and*

$$\int_{0^+} \frac{ds}{F_1(s)^{1/p}} = +\infty, \tag{2.8}$$

then $N(u(\cdot,t))$ *is empty.*

The proof is a direct consequence of [1, Corollary 1 and Theorem 1.20]. We send the reader to the general exposition made in [1] for more details and many other references dealing with the mentioned qualitative properties of the associated auxiliary stationary problems.

Remark 2.5. As the free boundary results for stationary problems are obtained in [1] through the *theory of local super and subsolutions,* the above-mentioned conclusions for periodic solutions can be extended to the case of other boundary conditions. Many variants are possible: variational inequalities, nondivergential form equations, suitable coupled systems (as, for instance, the model associated to the thermistor), and so forth.

Remark 2.6. The monotonicity conditions assumed in (H_f) can be replaced by some other more general conditions. In that case, several periodic solutions may coexist but the existence of a periodic free boundary still can be ensured for some of them (in the line of the results of [48, 49]).

Remark 2.7. In the absence of any absorption term (i.e., when $f(u) \equiv 0$), the existence of a periodic free boundary can be alternatively explained through the presence of a suitable convection term in the equation (which is not the case of problem (P_b)). The case of the stationary solutions was presented in [1, Section 2.4, Chapter 2] (see also [2, Section 4, Chapter 1]). Concerning the case of periodic solutions, we will limit ourselves to present here a concrete example (arising in the periodic filtration in a porous medium, as formulated in [6], and so with appropriate boundary conditions of Neumann type and time periodic coefficients). Here the transport term (or, equivalently, the right hand side term g) is suitably coupled with some appropriate boundary conditions. In our opinion, this example points out a potential research for more general formulations but we will not follow this line in the rest of this paper. Consider the function

$$u(x,t) = ((x+l-\sin\omega t)_-)^2 = \begin{cases} 0 & \text{if } x+l \geqslant \sin\omega t, \\ (x+l-\sin\omega t)^2 & \text{if } x+l < \sin\omega t. \end{cases} \tag{2.9}$$

Then, it is easy to check that u is the unique periodic solution of the problem

$$\begin{gathered} u_t = \varphi(u)_{xx} + \psi(t,x,u)_x \quad \text{in } (-l,0)\times\mathbb{R}, \\ -\varphi(u(0,t))_x - \psi(0,t,u(0,t)) = h(t)u(0,t) \quad t\in\mathbb{R}, \\ \varphi(u(-l,t))_x + \psi(-l,t,u(-l,t)) = g(t) \quad t\in\mathbb{R}, \\ u(x,t+T) = u(x,t) \quad \text{in } (-l,0)\times\mathbb{R}, \end{gathered} \tag{2.10}$$

where $T = 2\pi/\omega$, $\varphi(u) = u^2$,

$$\psi(t,x,u) = \begin{cases} 0 & \text{if } x+l \geqslant \sin\omega t, \\ -\omega\cos\omega t(x+l-\sin\omega t)^2 - 4(x+l-\sin\omega t)^3 & \text{if } x+l < \sin\omega t, \end{cases} \tag{2.11}$$

$h(t) = \omega\cos\omega t$, and

$$g(t) = \begin{cases} 0 & \text{if } \sin\omega t \leqslant 0, \\ -\omega\cos\omega t\sin^2\omega t & \text{if } 0 < \sin\omega t. \end{cases} \tag{2.12}$$

Obviously, the free boundary generated by such solution is the T-periodic function $x = -l + \sin\omega t$.

In the line of the precedent remarks, we will present now a result on the existence of the time periodic free boundary by adapting some of the energy methods developed since the beginning of the eighties for the study of nonlinear partial differential equations (see [2]). In that case a great generality is allowed in the formulation of the nonlinear equation. Consider for instance, the case of local (in space) solutions of the problem

$$(P^*)\begin{cases} \dfrac{\partial b(u)}{\partial t} - \operatorname{div}\mathbf{A}(x,t,u,Du) + B(x,t,u,Du) + C(x,t,u) = g & \text{in } B_\rho\times\mathbb{R}, \\ b(u(x,t+T)) = b(u(x,t)) & \text{in } B_\rho\times\mathbb{R}, \end{cases} \tag{2.13}$$

where $B_\rho = B_\rho(x_0)$ for some $x_0 \in \Omega$ and any $\rho \in [0,\rho_0]$, for some $\rho_0 > 0$. The general structural assumptions we will made are the following:

$$\begin{gathered}
|\mathbf{A}(x,t,r,\mathbf{q})|C_1|\mathbf{q}|^{p-1}, \qquad C_2\ |\mathbf{q}|^p \leqslant \mathbf{A}(x,t,r,\mathbf{q})\cdot\mathbf{q}, \\
|B(x,t,r,\mathbf{q})|C_3\ |r|^\alpha\ |\mathbf{q}|^\beta, \qquad C_0|r|^{q+1} \leqslant C(x,t,r)r, \\
C_6\ |r|^{\gamma+1} \leqslant G(r) \leqslant C_5\ |r|^{\gamma+1}, \quad \text{where } G(r) = b(r)r - \int_0^r b(\tau)d\tau,
\end{gathered} \tag{2.14}$$

with $b \in C(\mathbb{R})$ a nondecreasing function such that $b(0) = 0$. Here the possible time dependence of $\mathbf{A}$,B, and C is assumed to be T-periodic, and $C_1 - C_6$, p, α, β, σ, γ, k are positive constants.

Definition 2.8. A function $u(x,t)$, with $b(u) \in C([0,T] : L^1_{\text{loc}}(B_\rho))$, is called a local weak solution of the above problem if $b(u(x,t+T)) = b(u(x,t))$ in $B_\rho \times \mathbb{R}$; for any domain $\Omega' \subset \mathbb{R}^N$ with $\overline{\Omega'} \subset B_\rho$ one has $u \in L^\infty(0,T;\ L^{\gamma+1}(\Omega')) \cap L^p(0,T;\ W^{1,p}(\Omega'))$, $\mathbf{A}(\cdot,\cdot,u,Du)$, $B(\cdot,\cdot,u,Du)$, $C(\cdot,\cdot,u) \in L^1(B_\rho \times \mathbb{R})$, and for every test function $\varphi \in L^\infty(0,T;W^{1,p}(B_\rho)) \cap W^{1,2}(0,T;L^\infty(B_\rho))$ with $\varphi(x,t+T) = \varphi(x,t)$ in $B_\rho \times \mathbb{R}$ and for any $t \in [0,T]$ one has

$$\int_0^t \int_{B_\rho} \{b(u)\varphi_t - \mathbf{A}\cdot D\varphi - B\varphi - C\varphi\} dx\, dt - \int_\Omega b(u)\varphi dx \bigg|_0^t = -\int_0^t \int_{B_\rho} g\varphi dx\, dt. \tag{2.15}$$

As in [2, (see Section 4 of Chapter 4)] we will use some energy functions defined on domains of a special form. Given the nonnegative parameters ϑ and υ, we define the *energy set*

$$P(t) \equiv P(t;\vartheta,\upsilon) = \left\{(x,s) \in: |x-x_0| < \rho(s) \equiv \vartheta(s-t)^\upsilon,\ s \in (t,T]\right\}. \tag{2.16}$$

The shape of $P(t)$, *the local energy set,* is determined by the choice of the parameters ϑ and υ. We define the *local energy functions*

$$\begin{gathered}
E(P) := \int_{P(t)} |Du(x,\tau)|^p dx\, d\tau, \qquad C(P) := \int_{P(t)} |u(x,\tau)|^{q+1} dx\, d\tau \\
\Lambda(T) := \operatorname{ess}\sup_{s\in(t,T)} \int_{|x-x_0|<\vartheta(s-t)^\upsilon} |u(x,s)|^{\gamma+1} dx.
\end{gathered} \tag{2.17}$$

Although our results have a local nature (they are independent of the boundary conditions), it is useful to introduce some global information as, for instance, the one represented by *the global energy function*

$$D(u(\cdot,\cdot)) := \operatorname{ess}\sup_{s\in(0,T)} \int_\Omega |u(x,s)|^{\gamma+1} dx + \int_Q \left(|Du|^p + |u|^{q+1}\right) dx\, dt. \tag{2.18}$$

We assume the following conditions:

$$\begin{gathered} q < \gamma, \quad 1 + q < \frac{\gamma p}{p-1}, \\ g(x,t) \equiv 0 \quad \text{on } B_\rho(x_0), \quad \text{a.e. } t \in \mathbb{R} \end{gathered} \tag{2.19}$$

(recall that since we are dealing with local solutions, a global data $g(x,t)$ may be different than zero outside $B_\rho(x_0)$). In the presence of the first-order term, $B(\cdot,\cdot,u,Du)$, we will need the extra conditions

$$\begin{gathered} \alpha = \gamma - (1+\gamma)\beta/p, \\ C_3 < \left(C_0 \frac{p}{p-1}\right)^{(p-\beta)/p} \left(C_2 \frac{p}{\beta}\right)^{\beta/p} \quad \text{if } 0 < \beta < p, \\ C_3 < C_0 \quad \text{if } \beta = 0 \ (\text{resp. } C_0 < C_2 \text{ if } \beta = p). \end{gathered} \tag{2.20}$$

The next result shows the existence of a free boundary in a local way.

Theorem 2.9. *Any periodic weak solution satisfies that $u(x,t) \equiv 0$ on $B_{\rho_*} \times \mathbb{R}$, for some suitable $\rho_* \in (0,\rho_0)$, assumed that the global energy $D(u)$ is small enough.*

The proof of Theorem 2.3 follows the same lines of the proof of [2, Theorem 4.1]. Here we will only comment the different parts of it and the additional arguments necessary to adapt the mentioned result to the setting of periodic weak solutions. As a matter of fact, it is enough to take as *energy set* the cylinder itself (i.e., $\vartheta = 0$ and $\upsilon = 0$) but since other complementary results can be derived for other choices of ϑ and υ (see Remark 3.5 below), we will keep this generality for some parts of the proof. The first step is the so-called *integration-by-parts formula*

$$\begin{aligned} i_1 + i_2 + i_3 + i_4 = &\int_{P\cap\{t=T\}} G(u(x,t))dx \\ &+ \int_P \mathbf{A}\cdot Dudx\, d\theta + \int_P Budx\, d\theta + \left(\int_P C_0\, |u|^{q+1} dx\, d\theta\right) \\ \leqslant &\int_{\partial_l P} n_x \cdot \mathbf{A}ud\Gamma\, d\theta + \int_{\partial_l P} n_\tau G(u(x,t))d\Gamma\, d\theta \\ &+ \int_{P\cap\{t=0\}} G(u(x,t))dx := j_1 + j_2 + j_3, \end{aligned} \tag{2.21}$$

where $\partial_l P$ denotes the lateral boundary of P, that is, $\partial_l P = \{(x,s) : |x - x_0| = \vartheta(s-t)^\upsilon, s \in (t,T)\}$,$d\Gamma$ is the differential form on the hypersurface $\partial_l P \cap \{t = \text{const}\}$, and n_x and n_τ are the components of the unit normal vector to $\partial_l P$. This inequality can be proved by taking the cutoff function

$$\zeta(x,\theta) := \psi_\varepsilon(|x-x_0|,\theta)\xi_k(\theta)\frac{1}{h}\int_\theta^{\theta+h} T_m(u(x,s))ds, \quad h>0, \tag{2.22}$$

as test function,where T_m is the truncation at the level m,

$$\xi_k(\theta) := \begin{cases} 1 & \text{if } \theta \in \left[t, T-\dfrac{1}{k}\right], \\ k(T-\theta) & \text{for } \theta \in \left[T-\dfrac{1}{k}, T\right], \\ 0 & \text{otherwise, } k \in \mathbb{N}, \end{cases} \qquad \psi_\varepsilon(|x-x_0|,\theta) := \begin{cases} 1 & \text{if } d > \varepsilon, \\ \dfrac{1}{\varepsilon} d & \text{if } d < \varepsilon, \\ 0 & \text{otherwise,} \end{cases} \tag{2.23}$$

with $d = \text{dist}((x,\theta), \partial_l P(t))$ and $\varepsilon > 0$.

The second step consists in to get a *differential inequality for some energy function*. We take here the choice $\vartheta = 0$ and $\upsilon = 0$ so that $P = B_\rho(x_0) \times [0,T]$ (which implies that $j_2 = 0$), and we apply the periodicity conditions. So $i_1 = j_3$, and we get that $i_2 + i_3 + i_4 \leqslant j_1$. The rest of the proof uses (as in the mentioned reference) the following interpolation inequality: if $0 \leqslant q \leqslant p-1$, then there exists $L_0 > 0$ such that for all $v \in W^{1,p}(B_\rho)$

$$\|v\|_{p,S_\rho} \leqslant L_0\left(\|\nabla v\|_{p,B_\rho} + \rho^\delta \|v\|_{q+1,B_\rho}\right)^{\tilde\theta}\left(\|v\|_{r,B_\rho}\right)^{1-\tilde\theta} \tag{2.24}$$

$r \in [1, 1+\gamma]$, $\tilde\theta = ((pN - r(N-1))/((N+1)p - Nr))$, $\delta = -(1 + (((p-1-q)/p(1+q))N)$. Then, by applying Hölder inequality (several times), we arrive to the following differential inequality for the energy function $Y(\rho) := E + C$:

$$Y^\varepsilon \leqslant c\frac{\partial Y}{\partial \rho}, \tag{2.25}$$

for some $\varepsilon \in (0,1)$, where c depends (in a continuous and increasing way) or $D(u)$. The analysis of this inequality leads to the result as it was shown in the mentioned reference.

Remark 2.10. The cases of the time periodic *obstacle problem* and *Stefan problem* can be also treated followingthe arguments presented in [50] (for the initial value problems) and by arguing as in the precedent result.

Remark 2.11. It seems possible to adapt the energy methods concerning suitable higher-order equations (see [3, Section 8 of Chapter 3]) in order to show the existence of a periodic free boundary for the time periodic problem associated to such type of equations but we will not enter here in the details.

3. Periodical Time Connection between Stationary Episodes and on Disconnected Free Boundaries

We start this section by constructing an example of a periodic and nonconstant free boundary associated to problem (P). To simplify the exposition we will assume that $n = 1$, $\Omega = (-L, L)$ and that $f(s) = |s|^{q-1}s$ with $q < p - 1$. Let us define the function

$$u(x,t) = C[|x| - \tau(t)]_+^{p/(p-1-q)}, \tag{3.1}$$

where $C > 0$ and τ is a Lipschitz continuous T-periodic function such that, $0 \leqslant \tau(t) \leqslant L$ $\forall t \in \mathbb{R}$. It is easy to check (see a similar computation for the n-dimensional case in [31, Lemma 1.6]) that this function u is a T-periodic solution of problem (P) with $h(\pm L, t) = C(L - \tau(t))^{p/(p-1-q)} > 0$ and

$$g(x,t) = \left(\lambda C^q - \frac{p}{(p-1-q)} C\tau'(t) - C^{p-1}\right)[|x| - \tau(t)]_+^{(pq)/(p-1-q)}. \tag{3.2}$$

Hence, $g(x,t) \geqslant 0$ if and only if

$$\tau'(t) \leqslant \frac{C^{q-1}\left(\lambda - C^{(p-1-q)}\right)(p-1-q)}{p}. \tag{3.3}$$

For instance, if we take

$$\tau(t) = \begin{cases} l_0 + \dfrac{(l_1 - l_0)t}{t_1} & \text{if } 0 \leqslant t \leqslant t_1, \\ l_1 & \text{if } t_1 \leqslant t \leqslant t_2, \\ l_1 + \dfrac{(l_0 - l_1)}{T - t_2}(t - t_2) & \text{if } t_2 \leqslant t \leqslant T, \end{cases} \tag{3.4}$$

for some l_0,l_1 nonnegative given constants, $0 \leqslant t_1 \leqslant t_2 \leqslant T$, then (3.3) holds if we assume that

$$\max\left\{\frac{(l_1 - l_0)}{t_1}, \frac{(l_0 - l_1)}{T - t_2}\right\} \leqslant \frac{C^{q-1}\left(\lambda - C^{(p-1-q)}\right)(p-1-q)}{p}. \tag{3.5}$$

Remark 3.1. Notice that choice (3.4) leads to a transient periodic solution of the parabolic problem (P) connecting (in a finite time) the stationary solutions of problems

$$\begin{aligned} -\Delta_p v + \lambda f(v) &= g^* \quad \text{in } \Omega, \\ u &= h^* \quad \text{on } \partial\Omega, \end{aligned} \tag{SP}$$

for the data

$$\begin{aligned} g^*(x) &= \left(\lambda C^q - \frac{p}{(p-1-q)} C \frac{(l_1 - l_0)}{t_1} - C^{p-1}\right) [|x| - l_0]_+^{pq/(p-1-q)}, \\ h^*(\pm L) &= C(L - l_0)^{p/(p-1-q)}, \\ g^*(x) &= \left(\lambda C^q - C^{p-1}\right) [|x| - l_1]_+^{pq/(p-1-q)}, \\ h^*(\pm L) &= C(L - l_1)^{p/(p-1-q)}. \end{aligned} \tag{3.6}$$

It is well known that this behavior is very exceptional: for instance, it cannot hold in the case of linear parabolic problems. In particular, this solution can be used for different purposes in the study of controllability problems (see, e.g., [51]).

Remark 3.2. In [52] some support properties for the solution of the problem

$$\begin{aligned} b(u)_t - \Delta u - a(x,t)u = 0 \quad &\text{in } Q, \\ \frac{\partial u}{\partial n}(x,t) = 0 \quad &\text{on } \Sigma, \\ u(x,t+T) = u(x,t) \quad &\text{in } Q \end{aligned} \tag{$P_{b,N}$}$$

are given for $b(u) = |u|^{1/m-1}u$ and $m > 1$, under the periodicity condition $a(x,t+T) = a(x,t)$ in Q, for some x-Hölder and t-Lipschitz continuous function $a(x,t)$. The authors show that any nonnegative periodic solution has a support which is independent on t. Moreover, they also prove that if the subset $\Omega^+ := \{x \in \Omega : \int_0^T a(x,t)dt > 0\}$ is nonempty then either $u > 0$ or $u \equiv 0$ in $(0,T] \times \Omega_k^+$, where Ω_k^+ denotes any connected component of Ω^+. What the precedent example shows is that the nature of the stationary free boundary associated to the above problem is not generic but very peculiar due to assumption made on coefficient $a(x,t)$ and the Neumann boundary condition.

We will end this section by showing that it is possible to construct nonnegative periodic solutions of (P_b) giving rise to *disconnected free boundaries*, that is, with free boundaries given by closed hypersurfaces of the space $\mathbb{R}^{n+1}$.

We start by constructing some time periodic x-independent solutions with a support strictly contained in $[0,T]$. To do that we need the additional condition

$$\int_{0^+} \frac{\mathrm{d}s}{f(b^{-1}(s))} < +\infty. \tag{3.7}$$

Given $\varsigma > 0$ and $t^* \in [0,T]$, let $w(t : \varsigma, t^*)$ be the unique solution of the Cauchy problem

$$\begin{aligned} b(w(t))' + \underline{\lambda} f(w) = 0 \quad &t > t^*, \\ w(t^*) = \varsigma. & \end{aligned} \tag{3.8}$$

We have that if $z(t) := b(w(t))$ then

$$\Psi(z(t)) = \Psi(\varsigma) - (t - t^*) \tag{3.9}$$

with

$$\Psi(\tau) := \int_0^{\tau} \frac{ds}{f(b^{-1}(s))} \quad \text{for any } \tau > 0. \tag{3.10}$$

Denoting $\eta(\theta) = \Psi^{-1}(\theta)$, thanks to assumption (3.7), we have that

$$w(t) = \begin{cases} \eta\left(\Psi(\varsigma) - \underline{\lambda}(t - t^*)\right) & \text{if } t \in \left[t^*, t^* + \dfrac{\Psi(\varsigma)}{\underline{\lambda}}\right], \\ 0 & \text{if } t > t^* + \dfrac{\Psi(\varsigma)}{\underline{\lambda}}. \end{cases} \tag{3.11}$$

We assume the data such that

$$t^* + \frac{\Psi(\varsigma)}{\underline{\lambda}} < T. \tag{3.12}$$

Finally we define

$$U(t) = \begin{cases} \underline{w}(t) & \text{if } t \in [0, t^*], \\ w(t : \varsigma, t^*) & \text{if } t \in [t^*, T], \end{cases} \tag{3.13}$$

where $\underline{w} \in C([0, t^*])$ is such that $b(\underline{w})' \in L^1(0, t^*), \underline{w} \geqslant 0$, and $b(\underline{w}(t))' + \underline{\lambda} f(\underline{w}) \geqslant 0$ on $(0, t^*)$, and

$$\underline{w}(0) = 0, \qquad \underline{w}(t^*) = \varsigma. \tag{3.14}$$

Summarizing, we get the following.

Proposition 3.3. *Assume* (3.7)*. Let $\varsigma > 0$ and $t^* \in [0, T]$ such that* (3.12) *holds. Then the function $U(t)$ given by* (3.13) *is a nonnegative T-periodic solution of the problem*

$$\begin{gathered} b(w(t))' + \underline{\lambda} f(w) = g(t) \quad t \in \mathbb{R}, \\ b(w(t)) = b(w(t + T)) \quad t \in \mathbb{R}, \end{gathered} \tag{3.15}$$

where $g(t) \geqslant 0$ is the function given by

$$g(t) = \begin{cases} b\left(\underline{w}(t)\right)' + \underline{\lambda} f(\underline{w}) & \text{if } t \in [0, t^*], \\ 0 & \text{if } t \in [t^*, T]. \end{cases} \tag{3.16}$$

Some disconnected time periodic free boundaries can be formed under suitable conditions. The main idea is to put together the above two arguments and to consider the function

$$\overline{u}(x,t) = C[|x| - \tau(t)]_+^{p/(p-1-q)} + U(t). \tag{3.17}$$

It is a routine matter to check that $\overline{u}(x,t)$ is a T-periodic supersolution of the equation in (P) once we take $b(s) = s$ and $\underline{\lambda} = \lambda/2$, and we use the property that $f(a+b) \geqslant (1/2)f(a) + (1/2)f(b)$ for any $a,b \geqslant 0$ (which is consequence of the monotonicity of f). Analogously, since $U(t)$ is a subsolution of the equation, a careful choice of the auxiliary parameters and the application of the comparison principle lead to the following result:

Theorem 3.4. *Assume $\Omega = (-L, L)$, $f(s) = |s|^{q-1}s$ with $q < \min(1, p-1)$. Let $u(x,t)$ be the unique T-periodic solution of problem (P) corresponding to data $h(\pm L, t)$ and $g(x,t)$*

$$\begin{gathered} 0 \leqslant U(t) \leqslant h(\pm L, t) \leqslant C(L - \tau(t))^{p/(p-1-q)} + U(t), \\ G(t) \leqslant g(x,t) \leqslant \left(\frac{\lambda}{2}C^q - \frac{p}{(p-1-q)}C\tau'(t) - C^{p-1}\right)[|x| - \tau(t)]_+^{pq/(p-1-q)} + G(t), \end{gathered} \tag{3.18}$$

with $\tau(t)$ given by (3.4) *with $0 \leqslant l_0 < l_1 = L$, $0 < t_1 < t_2 < T$ and $C > 0$ such that*

$$\frac{(l_1 - l_0)}{t_1} \leqslant C^q\left(\frac{\lambda}{2} - C^{(p-1-q)}\right), \tag{3.19}$$

$U(t)$ given by (3.13) *with $\varsigma > 0$ and $t^* \in (t_1, t_2)$,$b(s) = s$ and $\underline{\lambda} = \lambda/2$,$\underline{w}(t) = 0$ if $t \in [0, t_1]$,*

$$G(t) = \begin{cases} 0 & \text{if } t \in [0, t_1], \\ \underline{w}'(t) + \dfrac{\lambda}{2}f(\underline{w}) & \text{if } t \in [t_1, t^*], \\ 0 & \text{if } t \in [t^*, t_2], \\ 0 & \text{if } t \in [t_2, T]. \end{cases} \tag{3.20}$$

Finally one takes

$$t_2 = t^* + \frac{2\Psi(\varsigma)}{\lambda}. \tag{3.21}$$

Then $U(t) \leqslant u(x,t) \leqslant C[|x| - \tau(t)]_+^{p/(p-1-q)} + U(t)$ *on* $\Omega \times \mathbb{R}$. *In particular the null set* $\cup_{t\in[0,T]} N(u(\cdot,t))$ *has at least two connected components since it contains the set*

$$\left\{(x,t) \in (-L,L) \times [0,t_1] : |x| \leqslant l_0 - \frac{l_0 t}{t_1}\right\} \cup \left\{(x,t) \in (-L,L) \times [t_2,T] : |x| \leqslant \frac{l_1}{T-t_2}(t-t_2)\right\}, \tag{3.22}$$

and $u(x,t) > 0$ *on the set* $(-L,L) \times (t_1,t_2)$.

Remark 3.5. It is possible to apply the above arguments to get the existence of a periodic free boundary in the special case of $h(x,t) \equiv 0$ on $\Sigma = \partial\Omega \times \mathbb{R}$ and with support of $g(,.t)$ strictly contained in $\Omega \times (0,T)$ if $t \in [0,T]$ (and then prolonged by T-periodicity to the whole domain $Q := \Omega \times \mathbb{R}$). In this way the support of the solution u is not connected but formed by periodical disconnected compact subsets of $\Omega \times \mathbb{R}$.

Remark 3.6. It seems possible to apply the energy method presented in the above section (but with *the local energy set* $P(t)$ with different shapes, that is, for different choices of the parameters ϑ and υ).

Acknowledgment

The research of the second author was partially supported by Project no. MTM200806208 of the DGISPI (Spain) and the Research Group MOMAT (no. 910480) supported by UCM. His research has received funding from the ITN "FIRST" of the Seventh Framework Programme of the European Community's (Grant agreement no. 238702).

References

[1] J. I. Diaz, *Nonlinear Partial Differential Equations and Free Boundaries*, Pitman, London, UK, 1985.
[2] S. N. Antontsev, J. I. Diaz, and S. Shmarev, *Energy Methods for Free Boundary Problems. Applications to Nonlinear PDEs and Fluid Mechanics*, vol. 48 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, Boston, Mass, USA, 2002.
[3] C. Bandle, R. P. Sperb, and I. Stakgold, "Diffusion and reaction with monotone kinetics," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 8, no. 4, pp. 321–333, 1984.
[4] C. Bandle and I. Stakgold, "The formation of the dead core in parabolic reaction-diffusion problems," *Transactions of the American Mathematical Society*, vol. 286, no. 1, pp. 275–293, 1984.
[5] A. S. Kalašnikov, "The nature of the propagation of perturbations in problems of nonlinear heat conduction with absorption," *Žurnal Vyčislitel' noĭ Matematiki i Matematičeskoĭ Fiziki*, vol. 14, pp. 891–905, 1974.
[6] M. Badii and J. I. Diaz, "Periodic solutions of a quasilinear parabolic boundary value problem arising in unsaturated flow through a porous medium," *Applicable Analysis*, vol. 56, no. 3-4, pp. 279–301, 1995.
[7] M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, New York, NY, USA, 1964.
[8] G. M. Lieberman, "Time-periodic solutions of quasilinear parabolic differential equations. I. Dirichlet boundary conditions," *Journal of Mathematical Analysis and Applications*, vol. 264, no. 2, pp. 617–638, 2001.
[9] R. Caşcaval and I. I. Vrabie, "Existence of periodic solutions for a class of nonlinear evolution equations," *Revista Matemática de la Universidad Complutense de Madrid*, vol. 7, no. 2, pp. 325–338, 1994.
[10] M. Badii and J. I. Diaz, "Time periodic solutions for a diffusive energy balance model in climatology," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 2, pp. 713–729, 1999.

[11] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non-Linéaires*, Dunod, Paris, France, 1969.

[12] T. I. Seidman, "Periodic solutions of a non-linear parabolic equation," *Journal of Differential Equations*, vol. 19, no. 2, pp. 242–257, 1975.

[13] J. Mawhin, "Periodic solutions of systems with p-Laplacian-like operators," in *Nonlinear Analysis and Its Applications to Differential Equations (Lisbon, 1998)*, vol. 43 of *Progress in Nonlinear Differential Equations and Their Applications.*, pp. 37–63, Birkhäuser, Boston, Mass, USA, 2001.

[14] W. Yifu, Y. Jingxue, and W. Zhuoqun, "Periodic solutions of evolution p-Laplacian equations with nonlinear sources," *Journal of Mathematical Analysis and Applications*, vol. 219, pp. 76–96, 1988.

[15] N. Mizoguchi, "Periodic solutions for degenerate diffusion equations," *Indiana University Mathematics Journal*, vol. 44, no. 2, pp. 413–432, 1995.

[16] N. Kenmochi, D. Kröner, and M. Kubo, "Periodic solutions to porous media equations of parabolic-elliptic type," *Journal of Partial Differential Equations*, vol. 3, no. 3, pp. 63–77, 1990.

[17] M. Kubo and N. Yamazaki, "Periodic stability of elliptic-parabolic variational inequalities with time-dependent boundary double obstacles," *Discrete and Continuous Dynamical Systems A*, pp. 614–623, 2007.

[18] W. Liu, "Periodic solutions of evolution m-Laplacian equations with a nonlinear convection term," *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 27368, 10 pages, 2007.

[19] F. E. Browder, "Nonlinear maximal monotone operators in Banach space," *Mathematische Annalen*, vol. 175, pp. 89–113, 1968.

[20] F. E. Browder, "Periodic solutions of nonlinear equations of evolution in infinite dimensional spaces," in *Lectures in Differential Equations, Vol. I*, A. K. Aziz, Ed., pp. 71–96, Van Nostrand, New York, NY, USA, 1969.

[21] Ph. Benilan and H. Brezis, "Solutions faibles d'équations d'évolution dans les espaces de Hilbert," *Université de Grenoble. Annales de l'Institut Fourier*, vol. 22, no. 2, pp. 311–329, 1972.

[22] H. Brezis, *Operateurs Maximaux Monotones et Semigroupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, The Netherlands, 1973.

[23] Ph. Bénilan, *Équation d'Évolution dans un Espace de Banach Quelconque et Applications*, thesis, Orsay, France, 1972.

[24] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International, Leyden, The Netherlands, 1976.

[25] P. Hess, *Periodic-Parabolic Boundary Value Problems and Positivity*, vol. 247 of *Pitman Research Notes in Mathematics Series*, Longman Scientific & Technical, Harlow, UK, 1991.

[26] D. Daners and P. Koch Medina, *Abstract Evolution Equations, Periodic Problems and Applications*, vol. 279 of *Pitman Research Notes in Mathematics Series*, Longman Scientific & Technical, Harlow, UK, 1992.

[27] B. A. Ton, "Periodic solutions of nonlinear evolution equations in Banach spaces," *Canadian Journal of Mathematics*, vol. 23, pp. 189–196, 1971.

[28] H. Amann, "Periodic solutions of semilinear parabolic equations," in *Nonlinear Analysis*, L. Cesari, R. Kannan, and H. Weinberger, Eds., pp. 1–29, Academic Press, New York, NY, USA, 1978.

[29] J. Prüss, "Periodic solutions of semilinear evolution equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 3, no. 5, pp. 601–612, 1979.

[30] J. Crema and J. L. Boldrini, "Periodic solutions of quasilinear equations with discontinuous perturbations," *Southwest Journal of Pure and Applied Mathematics*, no. 1, pp. 55–73, 2000.

[31] N. Alaa and M. Iguernane, "Weak periodic solutions of some quasilinear parabolic equations with data measures," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 3, article 46, 2002.

[32] N. Kenmochi and M. Kubo, "Periodic behavior of solutions to parabolic-elliptic free boundary problems," *Journal of the Mathematical Society of Japan*, vol. 41, no. 4, pp. 625–640, 1989.

[33] B. Kawhol and R. Rüll, "Periodic solutions of nonlinear heat equations under discontinuous boundary conditions," in *Equadiff 82*, vol. 1017 of *Lecture Notes in Mathematics*, pp. 322–327, Springer, New York, NY, USA, 1990.

[34] C. V. Pao, "Periodic solutions of parabolic systems with nonlinear boundary conditions," *Journal of Mathematical Analysis and Applications*, vol. 234, no. 2, pp. 695–716, 1999.

[35] M. Badii, "Periodic solutions for a nonlinear parabolic equation with nonlinear boundary conditions," *Rendiconti del Seminario Matematico. Università e Politecnico Torino*, vol. 67, no. 3, pp. 341–349, 2009.

[36] M. Badii, "Periodic solutions for semilinear parabolic problems with nonlinear dynamical boundary condition," *Rivista di Matematica della Università di Parma. Serie 7*, vol. 5, pp. 57–67, 2006.

[37] Y. Giga and N. Mizoguchi, "On time periodic solutions of the Dirichlet problem for degenerate parabolic equations of nondivergence type," *Journal of Mathematical Analysis and Applications*, vol. 201, no. 2, pp. 396–416, 1996.
[38] E. N. Dancer and P. Hess, "On stable solutions of quasilinear periodic-parabolic problems," *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV*, vol. 14, no. 1, pp. 123–141, 1987.
[39] P. Buonocore, "On the principal eigenvalue of a periodic-parabolic problem," *Ricerche di Matematica*, vol. 40, no. 2, pp. 335–343, 1991.
[40] J. L. Boldrini and J. Crema, "On forced periodic solutions of superlinear quasi-parabolic problems," *Electronic Journal of Differential Equations*, no. 14, pp. 1–18, 1998.
[41] A. El Hachimi and A. Lamrani Alaoui, "Existence of stable periodic solutions for quasilinear parabolic problems in the presence of well-ordered lower and upper-solutions," in *Proceedings of the Fez Conference on Partial Differential Equations*, vol. 9 of *Electronic Journal of Differential Equations Conference*, pp. 117–126, Southwest Texas State University, San Marcos, Tex, USA.
[42] T. Godoy, J. Hernández, U. Kaufmann, and S. Paczka, "On some singular periodic parabolic problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 62, no. 6, pp. 989–995, 2005.
[43] J. J. Morgan and S. L. Hollis, "The existence of periodic solutions to reaction-diffusion systems with periodic data," *SIAM Journal on Mathematical Analysis*, vol. 26, no. 5, pp. 1225–1232, 1995.
[44] M. Badii, "A generalized periodic thermistor model," *Rendiconti del Seminario Matematico. Università e Politecnico Torino*, vol. 65, no. 3, pp. 353–364, 2007.
[45] M. Badii, "Existence of periodic solutions for the thermistor problem with the Joule-Thomson effect," *Annali dell'Universitá di Ferrara. Sezione VII. Scienze Matematiche*, vol. 54, no. 1, pp. 1–10, 2008.
[46] M. Badii, "Existence of periodic solutions for the quasi-static thermoelastic thermistor problem," *Nonlinear Differential Equations and Applications*, vol. 16, no. 1, pp. 1–15, 2009.
[47] M. Badii, "The obstacle thermistor problem with periodic data," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 121, pp. 145–159, 2009.
[48] J. I. Diaz and J. Hernández, "Global bifurcation and continua of nonnegative solutions for a quasilinear elliptic problem," *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique*, vol. 329, no. 7, pp. 587–592, 1999.
[49] J. I. Diaz, J. Hernández, and F. J. Mancebo, "Branches of positive and free boundary solutions for some singular quasilinear elliptic problems," *Journal of Mathematical Analysis and Applications*, vol. 352, no. 1, pp. 449–474, 2009.
[50] J. I. Diaz, "Estimates of the location of a free boundary for the obstacle and Stefan problems obtained by means of some energy methods," *Georgian Mathematical Journal*, vol. 15, no. 3, pp. 475–484, 2008.
[51] J.-M. Coron, *Control and Nonlinearity*, vol. 136 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 2007.
[52] P. Hess, M. A. Pozio, and A. Tesei, "Time periodic solutions for a class of degenerate parabolic problems," *Houston Journal of Mathematics*, vol. 21, no. 2, pp. 367–394, 1995.